Complete Reference for Graceful Labeling of Cyclic Snakes

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Abstract:

A graph is called graceful if it has an graceful labeling. Rosa [1] introduced the idea of α- valuation, and we now known this as the graceful labeling. Here we will define the graph (m, k) C_4-snake and prove that the graphs kC_4- snake is graceful, we will prove the graph $(2, k)$ C_4-snake is graceful, we will introduce the graceful labeling of the graph $(3, k)$ C_4-snake, we will introduce the graceful labeling of the graph (m, k) C_4-snake**.** Here we will define the graph (m, k) C_8-snake and prove that the graphs kC_8- snake is graceful, we will prove the graph $(2, k)$ C_8-snake is graceful, we will introduce the graceful labeling of the graph $(3, k)$ C_8-snake, we will introduce the graceful labeling of the graph (m, k) C_8-snake**.**

Introduction:

A graph is called graceful if it has a graceful labeling. Rosa [1] introduced the idea of α- valuation, and we now known this as the graceful labeling. We can consider α- valuation as function ϕ. This function ϕ is a graceful valuation of a graph G with q edges if ϕ is an injection from the vertices of G to the set $\{0, 1, 2\ldots q\}$ such that, when each edge $\{u, v\}$ is assigned the label $k = |\phi(u) - \phi(v)|$ $1\leq k \leq q$.

The graphs considered here will be finite, undirected and simple. We denote the vertex sets and the edges sets of a graph G by $V(G)$ and $E(G)$ respectively.

A cycle in a graph G is a closed walk of the form $w_1w_2w_3\ldots w_k$ where $k \geq 3$, and if for some $i \neq j$ we have $w_i = w_j$ then $\{i, j\} = \{1, 2\}$. A kC_n-snake is a connected graph with k blocks; each of the blocks is isomorphic to the cycle C_n, such that the block-cut-vertex graph is a path. We also call a kC_n-snake as a cyclic snake. The graph kC_n-snake was introduced by Barrientos as generalization of the concept of triangular snake introduced by Rosa [2]. Rosa [2] The cycle C_n is graceful if and only if $n \equiv 0$ or 3 (mod 4), E. M. Badr, M. I. Moussa and K. Kathiresan [3] proved that The cycle C_n is odd graceful if n is even, $n \geq 4$. Now we define The graphs mC_4 as the family of graphs consisting of m copies of C_4 with two non adjacent vertices in common

Example 1

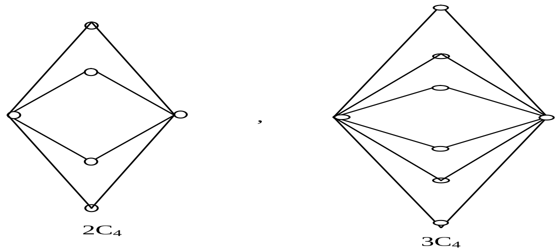

Figure1: The graphs $2C_4$ & $3C_4$.

and The graphs (m, k) C_4 as the family of graphs kC_4-snake where every block has m copies of C_4 with two non adjacent vertices in common. such that the number of blocks is denoted by k.

Example 2

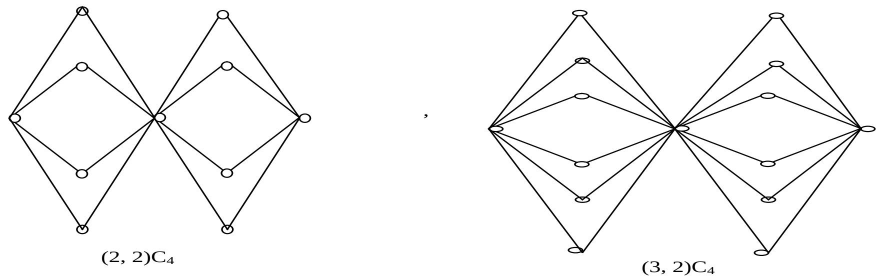

Figure 2: The graphs (2, 2) C_4 & (3,2) C_4.

Main Results:

Theorem 1: The graph kC_4-snake " (1, k) C_4-snake" is graceful.

Proof

Let kC_4-snake be a graph and has q edges. Let $u_1u_2u_3\ldots u_{k+1}$, $w_{11}w_{12}w_{21}\ldots w_{k1}w_{k2}$ are the vertices of kC_4-snake, such that w_{ij} are put between u_i and u_{i+1}, $i = 1, 2, 3\ldots k$, $j = 1, 2$, w_{ij} is above w_{ij+1} where $j = 1, 2$, the graph kC_4-snake has number of edges "q" $4k$, as shown in the next figure.

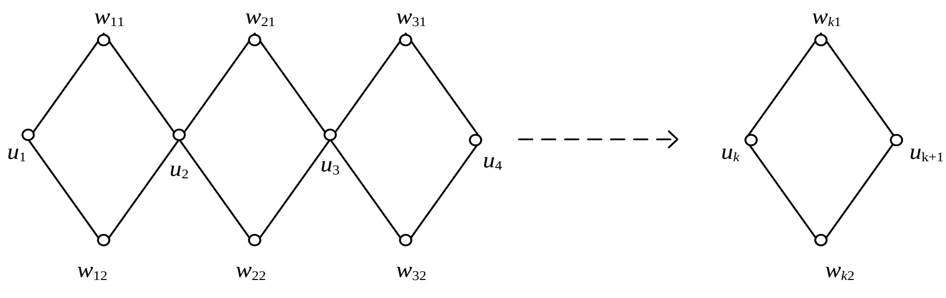

Figure 3: The graph kC_4-snake.

The number of edges" q"= $4k$

Define $\phi: V(G) \to \{0, 1, 2\ldots q\}$ as following:

$\phi(u_i) = 2i - 2$, $i = 1, 2, 3\ldots k + 1$

$\phi(w_{ij}) = q - 2i - j + 3$, $i = 1, 2, 3\ldots k$, $j = 1, 2$

From the definition of ϕ we find:

• $\forall\ v \in V(G)$: $V(G)$ is a set contains all the vertices of the graph $G = kC_4$-snake

$$\exists \max_{v \in V(G)} \phi(v) = \max\{ \max_{1 \le i \le k+1} (2i-2), \max_{\substack{1\le i \le k,\\ j=1,2}} (q - 2i - j + 3)\}$$

$= \max\{\max\{0, 2, 4 \ldots 2k\}, \max_{j=1,2} \{q - j + 1, q - j - 1, q - j - 3 \ldots q - j - 2k + 3\}\}$

$= \max\{2k, \max\{q, q - 2, q - 4 \ldots q - 2k + 2, q - 1, q - 3 \ldots q - 2k + 1\}\}$

$= \max\{2k, q\} \quad : q = 4k$

$= q$

Hence, we find that :-

$$\phi(v) \in \{0, 1, 2 \ldots q - 1, q\}$$

• It's obvious that $\forall\, u, v \in V(G) \ \exists\ \phi(u) \neq \phi(v) \in \{0, 1, 2 \ldots q - 1, q\}$

$\therefore$ The function ϕ is one-to-one mapping from the vertex set of G "$V(G)$" to the set $\{0, 1, 2 \ldots q\}$

• Now, we want to show that the labels of the edges of G belong to the set $\{1, 2 \ldots q\}$ and that's as following:

The range of $|\phi(w_{i1}) - \phi(u_i)| = \{q - 4i + 4 \quad , i = 1, 2 \ldots k\}$

$= \{q, q - 4, q - 8 \ldots q - 4k + 4\} \quad , q = 4k$

$= \{q, q - 4, q - 8 \ldots 4\}$

The range of $|\phi(w_{i1}) - \phi(u_{i+1})| = \{q - 4i + 2 \quad , i = 1, 2 \ldots k\}$

$= \{q - 2, q - 6 \ldots q - 4k + 2\} \quad , q = 4k$

$= \{q - 2, q - 6 \ldots 2\}$

The range of $|\phi(w_{i2}) - \phi(u_i)| = \{q - 4i + 3 \quad , i = 1, 2 \ldots k\}$

$= \{q - 1, q - 5 \ldots q - 4k + 3\} \quad , q = 4k$

$= \{q - 1, q - 5 \ldots 3\}$

The range of $|\phi(w_{i2}) - \phi(u_{i+1})| = \{q - 4i + 1 \quad , i = 1, 2 \ldots k\}$

$= \{q - 3, q - 7 \ldots q - 4k + 1\} \quad , q = 4k$

$= \{q - 3, q - 7 \ldots 1\}$

Hence, $\{|\phi(u) - \phi(v)| : uv \in E(G)\} = \{1, 2, 3 \ldots q\}$.

So the graph kC_4-snake is graceful.

Example 3

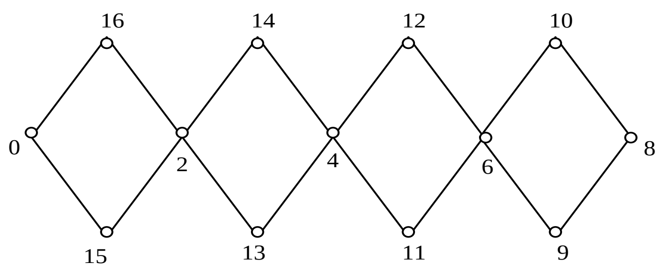

Figure 4: The graceful labeling of the graph $4C_4$-snake

Theorem 2: The graph (2, k) C_4-snake is graceful.

Proof

Let u_i, where i = 1, 2, 3… k + 1, w_{ij}, where i = 1, 2, 3… k, j = 1, 2, 3, 4 are the vertices of (2, k) C_4-snake, such that w_{ij} are put between u_i and u_{i+1}, i = 1, 2, 3… k, j = 1, 2, 3, 4, w_{ij} is above $w_{i(j+1)}$, where j = 1, 2, 3, 4, the graph (2, k) C_4-snake has number of edges "q" 8k, as shown in the next figure.

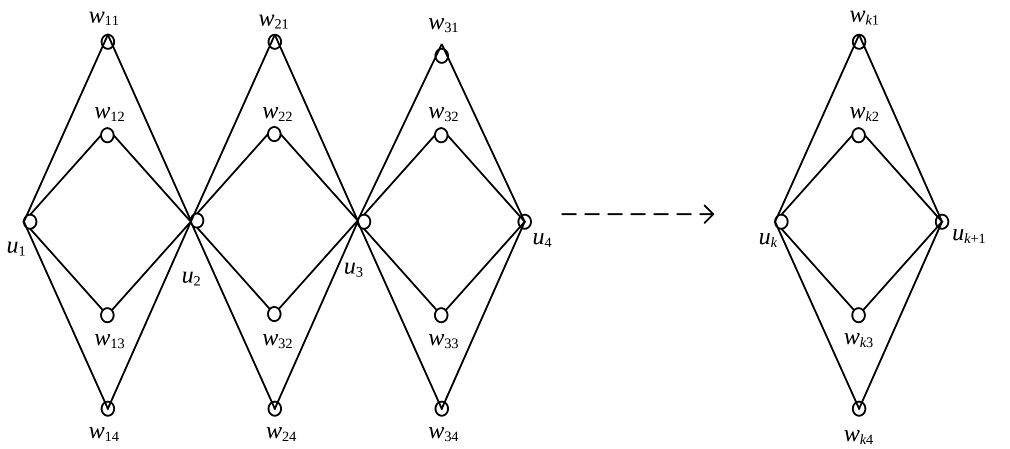

Figure 5: The graph (2, k) C_4-snake.

The number of edges" q" = 8k

Define ϕ: $V(G) \rightarrow \{0, 1, 2 \ldots q\}$ as following:

$\phi(u_i) = 4i - 4$, i =1, 2, 3… k + 1

$\phi(w_{ij}) = q - 4i - j + 5$, i =1, 2, 3… k , j =1, 2, 3, 4

From the definition of ϕ we find:

• $\forall\ v \in V(G)$: $V(G)$ is a set contains all the vertices of the graph G = (2, k) C_4-snake

$$\exists \max_{v\in V(G)} \phi(v) = \max\{ \max_{1\le i\le k+1}(4i-4), \max_{\substack{1\le i\le k,\\ 1\le j\le 4}}(q-4i-j+5)\}$$

$$= \max\{\max \{0, 4, 8 \ldots 4k\}, \max_{1\le j\le 4} \{q - j + 1, q - j - 3 \ldots q - j - 4k + 5\}\}$$
$$= \max\{4k, \max\{q, q - 1, q - 2, q - 3 \ldots q - 4k + 1\}\}$$
$$= \max\{4k, q \} \quad : q = 8k$$
$$= q$$

Hence, we find that :−

$$\phi(v) \in \{ 0, 1, 2 \ldots q - 1, q \}$$

• It's obvious that $\forall\ u, v \in V(G)\ \exists\ \phi(u) \neq \phi(v) \in \{0, 1, 2 \ldots q - 1, q\}$

$\therefore$ The function ϕ is one-to-one mapping from the vertex set of G "$V(G)$" to the set $\{ 0, 1, 2 \dots q \}$

• Now, we want to show that the labels of the edges of G belong to the set $\{1, 2 \dots q \}$ and that's as following:

The range of $| \phi(w_{i1}) - \phi(u_i) | = \{q - 8i + 8 \quad , i = 1, 2 \dots k \}$

$= \{ q, q - 8, q - 16 \dots q - 8k + 8 \} \quad , q = 8k$

$= \{ q, q - 8, q - 16 \dots 8\}$

The range of $| \phi(w_{i1}) - \phi(u_{i+1}) | = \{ q - 8i + 4 \quad , i = 1, 2 \dots k\}$

$= \{ q - 4, q - 12 \dots q - 8k + 4\} \quad , q = 8k$

$= \{ q - 4, q - 12 \dots 4\}$

The range of $| \phi(w_{i2}) - \phi(u_i) | = \{q - 8i + 7 \quad , i = 1, 2 \dots k \}$

$= \{ q - 1, q - 9 \dots q - 8k + 7\} \quad , q = 8k$

$= \{ q - 1, q - 9 \dots 7\}$

The range of $| \phi(w_{i2}) - \phi(u_{i+1}) | = \{ q - 8i + 3 \quad , i = 1, 2 \dots k \}$

$= \{q - 5, q - 13 \dots q - 8k + 3\} \quad , q = 8k$

$= \{ q - 5, q - 13 \dots 3\}$

The range of $| \phi(w_{i3}) - \phi(u_i) | = \{q - 8i + 6 \quad , i = 1, 2 \dots k \}$

$= \{ q - 2, q - 10 \dots q - 8k + 6 \} \quad , q = 8k$

$= \{ q - 2, q - 10 \dots 6\}$

The range of $| \phi(w_{i3}) - \phi(u_{i+1}) | = \{ q - 8i + 2 \quad , i = 1, 2 \dots k\}$

$= \{ q - 6, q - 14 \dots q - 8k + 2\} \quad , q = 8k$

$= \{ q - 6, q - 14 \dots 2\}$

The range of $| \phi(w_{i4}) - \phi(u_i) | = \{q - 8i + 5 \quad , i = 1, 2 \dots k \}$

$= \{ q - 3, q - 11 \dots q - 8k + 5\} \quad , q = 8k$

$= \{ q - 3, q - 11 \dots 5\}$

The range of $| \phi(w_{i4}) - \phi(u_{i+1}) | = \{ q - 8i + 1 \quad , i = 1, 2 \dots k \}$

$= \{q - 7, q - 15 \dots q - 8k + 1\} \quad , q = 8k$

$= \{ q - 7, q - 15 \dots 1\}$

Hence, $\{ | \phi(u) - \phi(v) | : u\, v \in E(G) \} = \{ 1, 2, 3 \dots q\}$.

So the graph $(2, k)$ C_4-snake is graceful.

Example 4

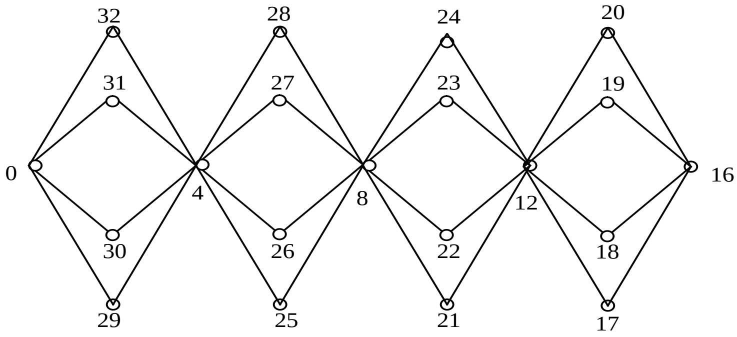

Figure 6: The graceful labeling of the graph (2, 4) C_4-snake

Theorem 3: The graph (3, k) C_4-snake is graceful.

Proof

Let u_i, where i = 1, 2, 3… k + 1, w_{ij} where i = 1, 2, 3… k, j = 1, 2, 3, 4, 5, 6 are the vertices of (3, k) C_4-snake, such that w_{ij} are put between u_i and u_{i+1}, i = 1, 2, 3… k, j = 1, 2, 3, 4, 5, 6, w_{ij} is above $w_{i(j+1)}$, where j = 1, 2, 3, 4, 5, 6, the graph (3, k) C_4-snake has number of edges "q" 12k, as shown in the next figure.

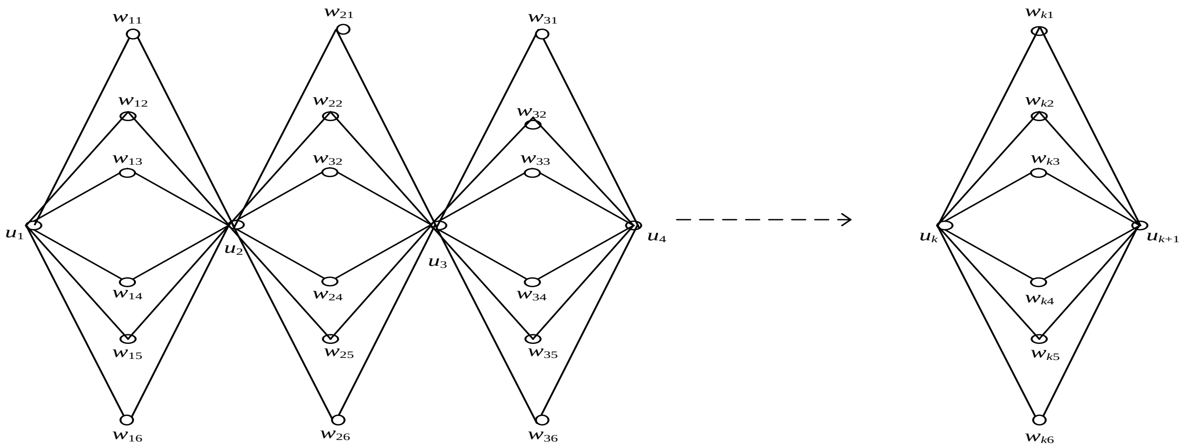

Figure 7: The graph (3, k) C_4-snake.

The number of edges" q" = 12k

Define $\phi : V(G) \to \{0, 1, 2 \ldots q\}$ as following:

$\phi(u_i) = 6i - 6$, i =1, 2, 3… k + 1

$\phi(w_{ij}) = q - 6i - j + 7$, i =1, 2, 3… k , j=1, 2, 3, 4, 5, 6

From the definition of ϕ we find:

• $\forall\ v \in V(G)$: $V(G)$ is a set contains all the vertices of the graph G = (3, k) C_4-snake

$\exists \max_{v \in V(G)} \phi(v) = \max\{ \max_{1 \le i \le k+1} (6i - 6), \max_{1 \le i \le k, 1 \le j \le 6} (q - 6i - j + 7)\}$

$= \max\{\max \{0, 6, 12 \ldots 6k\}, \max_{1 \le j \le 6} \{q - j + 1, q - j - 5 \ldots q - j - 6k + 7\}\}$

$= \max\{6k, \max\{q, q - 1, q - 2, q - 3 \ldots q - 6k + 1\}\}$

$= \max\{6k, q\} \quad : q = 12k$

$= q$

Hence, we find that :-

$\phi(v) \in \{0, 1, 2 \ldots q - 1, q\}$

• It's obvious that $\forall\, u, v \in V(G) \ \exists\ \phi(u) \neq \phi(v) \in \{0, 1, 2 \ldots q - 1, q\}$

$\therefore$ The function ϕ is one-to-one mapping from the vertex set of G "$V(G)$" to the set $\{0, 1, 2 \ldots q\}$

• Now, we want to show that the labels of the edges of G belong to the set $\{1, 2 \ldots q\}$ and that's as following:

The range of $|\phi(w_{i1}) - \phi(u_i)| = \{q - 12i + 12 \quad , i = 1, 2 \ldots k\}$

$= \{q, q - 12, q - 24 \ldots q - 12k + 12\} \quad , q = 12k$

$= \{q, q - 12, q - 24 \ldots 12\}$

The range of $|\phi(w_{i1}) - \phi(u_{i+1})| = \{q - 12i + 6 \quad , i = 1, 2 \ldots k\}$

$= \{q - 6, q - 18 \ldots q - 12k + 6\} \quad , q = 12k$

$= \{q - 6, q - 18 \ldots 6\}$

The range of $|\phi(w_{i2}) - \phi(u_i)| = \{q - 12i + 11 \quad , i = 1, 2 \ldots k\}$

$= \{q - 1, q - 13 \ldots q - 12k + 11\} \quad , q = 12k$

$= \{q - 1, q - 13 \ldots 11\}$

The range of $|\phi(w_{i2}) - \phi(u_{i+1})| = \{q - 12i + 5 \quad , i = 1, 2 \ldots k\}$

$= \{q - 7, q - 19 \ldots q - 12k + 5\} \quad , q = 12k$

$= \{q - 7, q - 19 \ldots 5\}$

The range of $|\phi(w_{i3}) - \phi(u_i)| = \{q - 12i + 10 \quad , i = 1, 2 \ldots k\}$

$= \{q - 2, q - 14 \ldots q - 12k + 10\} \quad , q = 12k$

$= \{q - 2, q - 12 \ldots 10\}$

The range of $|\phi(w_{i3}) - \phi(u_{i+1})| = \{q - 12i + 4 \quad , i = 1, 2 \ldots k\}$

$= \{q - 12, q - 20 \ldots q - 12k + 4\} \quad , q = 12k$

$= \{ q - 12, q - 20 \ldots 4\}$

The range of $| \phi(w_{i4}) - \phi(u_i) | = \{q - 12i + 9 \quad , i = 1, 2 \ldots k \}$

$= \{ q - 3, q - 15 \ldots q - 12k + 9\} \quad , q = 12k$

$= \{ q - 3, q - 15 \ldots 9\}$

The range of $| \phi(w_{i4}) - \phi(u_{i+1}) | = \{ q - 12i + 3 \quad , i = 1, 2 \ldots k \}$

$= \{q - 9, q - 21 \ldots q - 12k + 3\} \quad , q = 12k$

$= \{ q - 9, q - 21 \ldots 3\}$

The range of $| \phi(w_{i5}) - \phi(u_i) | = \{q - 12i + 8 \quad , i = 1, 2 \ldots k \}$

$= \{ q - 4, q - 16 \ldots q - 12k + 8\} \quad , q = 12k$

$= \{ q - 4, q - 16 \ldots 8\}$

The range of $| \phi(w_{i5}) - \phi(u_{i+1}) | = \{ q - 12i + 2 \quad , i = 1, 2 \ldots k\}$

$= \{ q - 10, q - 22 \ldots q - 12k + 2\} \quad , q = 12k$

$= \{ q - 10, q - 22 \ldots 2\}$

The range of $| \phi(w_{i6}) - \phi(u_i) | = \{q - 12i + 7 \quad , i = 1, 2 \ldots k \}$

$= \{ q - 5, q - 17 \ldots q - 12k + 7\} \quad , q = 12k$

$= \{ q - 5, q - 17 \ldots 7\}$

The range of $| \phi(w_{i6}) - \phi(u_{i+1}) | = \{ q - 12i + 1 \quad , i = 1, 2 \ldots k \}$

$= \{q - 11, q - 23 \ldots q - 12k + 1\} \quad , q = 12k$

$= \{ q - 11, q - 23 \ldots 1\}$

Hence, $\{ | \phi(u) - \phi(v) | : u\,v \in E(G) \} = \{ 1, 2, 3 \ldots q\}$.

So the graph $(3, k)$ C_4-snake is graceful.

Example 5

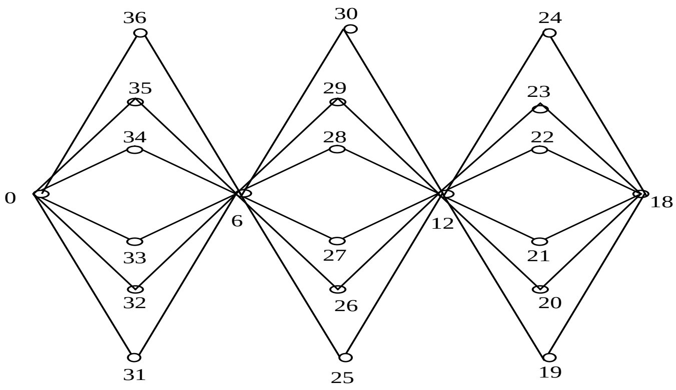

Figure 8: The graceful labeling of the graph (3, 3) C_4-snake

Theorem 4: The graph (*m*, *k*) C_4-snake is graceful.

Proof

Let u_i, where i = 1, 2, 3… k + 1, w_{ij}, where i = 1, 2, 3… k, j = 1, 2, 3… m are the vertices of (*m*, *k*) C_4-snake, such that w_{ij} are put between u_i and u_{i+1}, i = 1, 2, 3… k, j = 1, 2, 3… m, w_{ij} is above $w_{i(j+1)}$ where j = 1, 2, 3… m, the graph (*m*, *k*) C_4-snake has number of edges "*q*" 4*mk*, as shown in the next figure.

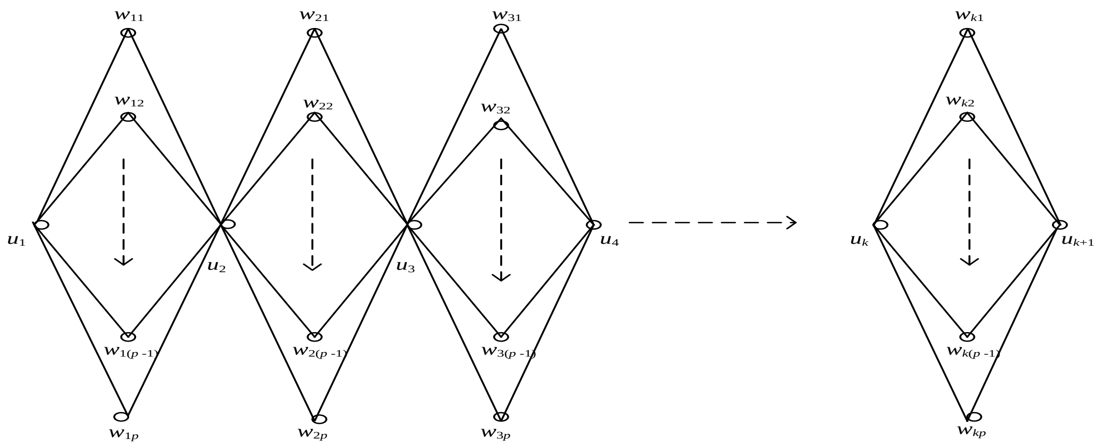

Figure 9: The graph (*m*, *k*) C_4-snake.

The number of edges" q" = 4mk , m = 1, 2, 3….. , p = 2m

Define $\phi : V(G) \rightarrow \{0, 1, 2 \ldots q\}$.

$\phi(u_i) = 2m(i-1)$, i = 1, 2, 3… k + 1

$\phi(w_{ij}) = q - 2m(i-1) - j + 1$, i = 1, 2, 3… k , j = 1, 2, 3…2m

From the definition of ϕ we find:

• $\forall\ v \in V(G)$: $V(G)$ is a set contains all the vertices of the graph G = (3, k) C_4-snake

$\exists \max_{v\in V(G)} \phi(v) = \max\{ \max_{1\le i \le k+1}(2mi - 2m), \max_{\substack{1\le i\le k, \\ 1\le j\le 2m}} (q - 2mi - j + 2m + 1)\}$

$= \max\{\max \{0, 2m, 4m \dots 2mk\}, \max_{1\le j\le 2m} \{q - j + 1, q - j - 2m + 1 \dots q - j - 2mk + 2m + 1\}\}$

$= \max\{2mk, q\} \quad : q = 4mk$

$= q$

Hence, we find that :−

$\phi(v) \in \{0, 1, 2 \dots q-1, q\}$

• It's obvious that $\forall\ u, v \in V(G) \ \exists\ \phi(u) \neq \phi(v) \in \{0, 1, 2 \dots q-1, q\}$

$\therefore$ The function ϕ is one-to-one mapping from the vertex set of G "$V(G)$" to the set $\{0, 1, 2 \dots q\}$

• Now, we want to show that the labels of the edges of G belong to the set $\{1, 2 \dots q\}$ and that's as following:

The range of $|\phi(w_{ij}) - \phi(u_i)| = \{q - 4mi - j + 4m + 1 \quad , i = 1, 2 \dots k, j = 1, 2, 3 \dots 2m\}$

$= \{q - j + 1, q - j - 4m + 1 \dots q - j - 4mk + 4m + 1 \quad , j = 1, 2 \dots 2m\}$

The range of $|\phi(w_{ij}) - \phi(u_i)| = \{q - 4mi - j + 2m + 1 \quad , i = 1, 2 \dots k, j = 1, 2, 3 \dots 2m\}$

$= \{q - j - 2m + 1, q - j - 6m + 1 \dots q - j - 4mk + 2m + 1, j = 1, 2 \dots 2m\}$

Hence, $\{ |\phi(u) - \phi(v)| : uv \in E(G)\} = \{1, 2, 3 \dots q\}$.

So the graph (m, k) C_4-snake is graceful.

Example 6

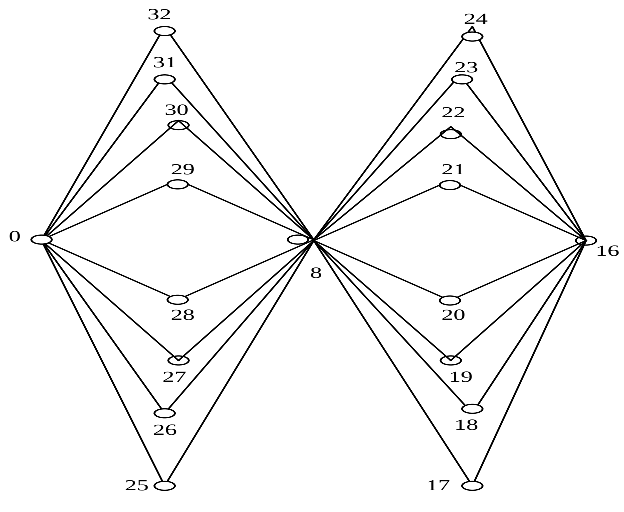

Figure 10: The graceful labeling of the graph (4, 2) C_4-snake

Theorem 1: The graph kC_8-snake " (1, k) C_8-snake" is graceful.

Proof

Let u_i, where i = 1, 2, 3… k + 1, & w_{ij} where i = 1, 2, 3… k, j = 1, 2, & x_{ij} where i = 1, 2, 3… 2k, j = 1, 2 are the vertices of kC_8-snake, such that w_{ij} is put between u_i and u_{i+1}, x_{pj} is put between w_{ij} and u_i where i = 1, 2, 3…k, p is odd(p = 1, 3…2k - 1), j= 1, 2, x_{pj} is put between w_{ij} and u_{i+1} where i = 1, 2, 3…k, p is even(p = 2, 4…2k), j = 1, 2, w_{ij} is above $w_{i(j+1)}$ where j = 1, 2, x_{ij} is above $x_{i(j+1)}$ where j = 1, 2, the graph $kC_8 \odot mk1$-snake has number of edges "q" 8k, as shown in the next figure.

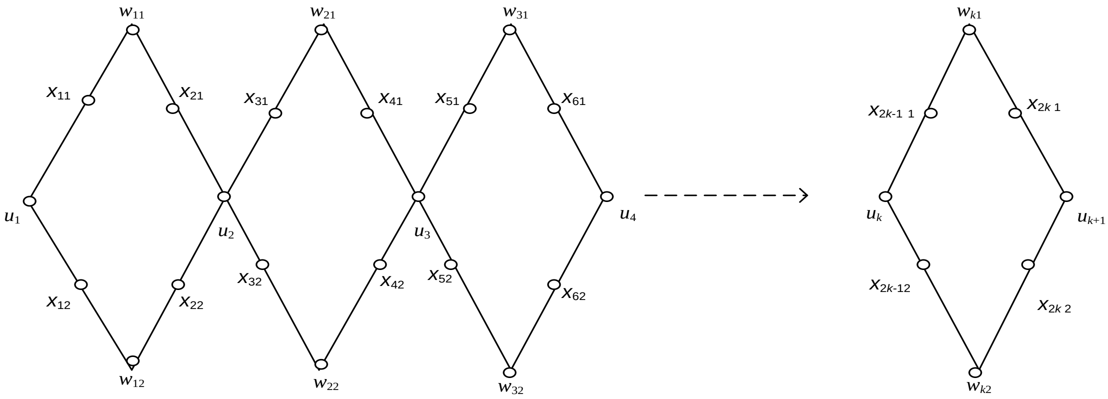

Figure 3: The graph kC_8.

The number of edges" q"= 8k

Define ϕ: $V(G) \rightarrow \{0, 1, 2\ldots q\}$ as following:

$\phi(u_i) = 4i - 4$, i= 1, 2, 3… k+1

$\phi(w_{ij}) = 4i - 2j + 1$, i= 1, 2, 3… k , j =1, 2

$\phi(x_{ij}) = q - 2i - j + 3$, i= 1, 2, 3… 2k for all j =1, 2

From the definition of ϕ we find:

• $\forall\ v \in V(G)$: $V(G)$ is a set contains all the vertices of the graph $G = kC_8$-snake

$$\exists \max_{v\in V(G)} \phi(v) = \max\{ \max_{1\le i\le k+1}(4i-4),\ \max_{\substack{1\le i\le k,\\ j=1,2}}(4i-2j+1),\ \max_{\substack{1\le i\le 2k,\\ j=1,2}}(q-2i-j+3)\}$$

$$= \max\{ \max\{0, 4, 8 \ldots 4k\},\ \max_{j=1,2}\{5-2j, 9-2j \ldots 4k-2j+1\},$$

$$\max_{j=1,2}\{q-j+1, q-j-1 \ldots q-j-4k+3\}\}$$

$$= \max\{4k, \max\{3, 1, 7, 5 \ldots 4k-1, 4k-3\}, \max\{q, q-1, q-2 \ldots q-4k+2, q-4k+1\}\}$$

$$= \max\{4k, 4k-1, q\}$$
$$= q$$

Hence, we find that :−

$$\phi(v) \in \{0, 1, 2 \ldots q-1, q\}$$

• It's obvious that $\forall\ u, v \in V(G)\ \exists\ \phi(u) \neq \phi(v) \in \{0, 1, 2 \ldots q-1, q\}$

$\therefore$ The function ϕ is one-to-one mapping from the vertex set of G "$V(G)$" to the set $\{0, 1, 2 \ldots q\}$

• Now, we want to show that the labels of the edges of G belong to the set $\{1, 2 \ldots q\}$ and that's as following:

The range of $|\phi(x_{i^*1}) - \phi(u_i)| = \{q - 2i^* - 4i + 6 \quad , i^* = 1, 3, 5 \ldots 2k-1, i = 1, 2 \ldots k\}$

$= \{q, q-8, q-16 \ldots q-8k+8\} \quad , q = 8k$

$= \{q, q-8, q-16 \ldots 8\}$

The range of $|\phi(x_{i^*1}) - \phi(w_{i1})| = \{q - 2i^* - 4i + 3 \quad , i^* = 1, 3, 5 \ldots 2k-1, i = 1, 2 \ldots k\}$

$= \{q-3, q-11, q-19 \ldots q-8k+5\} \quad , q = 8k$

$= \{q-3, q-11, q-19 \ldots 5\}$

The range of $|\phi(x_{i^*1}) - \phi(w_{i1})| = \{q - 2i^* - 4i + 3 \quad , i^* = 2, 4, 6 \ldots 2k, i = 1, 2 \ldots k\}$

$= \{q-5, q-13 \ldots q-8k+3\} \quad , q = 8k$

$= \{q-5, q-13 \ldots 3\}$

The range of $|\phi(x_{i^*1}) - \phi(u_{i+1})| = \{q - 2i^* - 4i + 2 \quad , i^* = 2, 4, 6 \ldots 2k, i = 1, 2 \ldots k\}$

$= \{q-6, q-14 \ldots q-8k+2\} \quad , q = 8k$

$= \{q-6, q-14 \ldots 2\}$

The range of $|\phi(x_{i^*2}) - \phi(u_i)| = \{q - 2i^* - 4i + 5 \quad , i^* = 1, 3, 5 \ldots 2k-1, i = 1, 2 \ldots k\}$

$= \{q-1, q-9 \ldots q-8k+7\} \quad , q = 8k$

$= \{q-1, q-9 \ldots 7\}$

The range of $|\phi(x_{i^*2}) - \phi(w_{i2})| = \{q - 2i^* - 4i + 4 \quad , i^* = 1, 3, 5 \ldots 2k-1, i = 1, 2 \ldots k\}$

$= \{q-2, q-10 \ldots q-8k+6\} \quad , q = 8k$

$= \{q-2, q-10 \ldots 6\}$

The range of $|\phi(x_{i^*2}) - \phi(w_{i2})| = \{q - 2i^* - 4i + 4 \quad , i^* = 2, 4, 6 \ldots 2k, i = 1, 2 \ldots k\}$

$= \{q-4, q-12 \ldots q-8k+4\} \quad , q = 8k$

$= \{q-4, q-12 \ldots 4\}$

The range of $|\phi(x_{i^*2}) - \phi(u_{i+1})| = \{q - 2i^* - 4i + 1 \quad , i^* = 2, 4, 6 \ldots 2k, i = 1, 2 \ldots k\}$

$= \{q-7, q-15 \ldots q-8k+1\} \quad , q = 8k$

$= \{ q - 7, q - 15 \ldots 1\}$

Hence, $\{ | \phi(u) - \phi(v) | : u\,v \in E(G) \} = \{ 1, 2, 3 \ldots q\}$.

So the graph kC_8 -snake is graceful.

Example 3

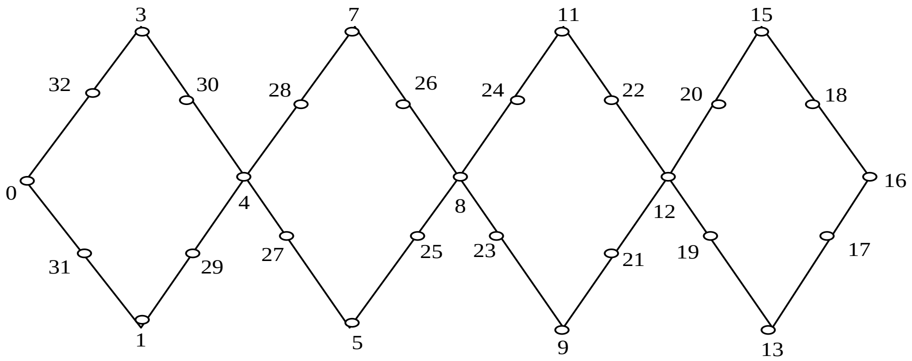

Figure 4: The graceful labeling of the graph 4 C_8-snake

Theorem 2: The graph (2, k) C_8-snake is graceful.

Proof

Let u_i, where i = 1, 2, 3… k + 1, & w_{ij} where i = 1, 2, 3… k, j = 1, 2, 3, 4, x_{ij} where i = 1, 2, 3… 2k, j = 1, 2, 3, 4 are the vertices of (2, k) C_8-snake, such that w_{ij} is put between u_i and u_{i+1}, x_{pj} is put between w_{ij} and u_i where i = 1, 2, 3…k, p is odd(p = 1, 3…2k - 1), j = 1, 2, 3, 4, x_{pj} is put between w_{ij} and u_{i+1} where i = 1, 2, 3…k, p is even(p = 2, 4…2k), j = 1, 2, 3, 4, w_{ij} is above $w_{i(j+1)}$ where j = 1, 2, 3, 4, x_{ij} is above $x_{i(j+1)}$ where j = 1, 2, 3, 4, the graph (2, k) C_8-snake has number of edges "q" 16k, as shown in the next figure.

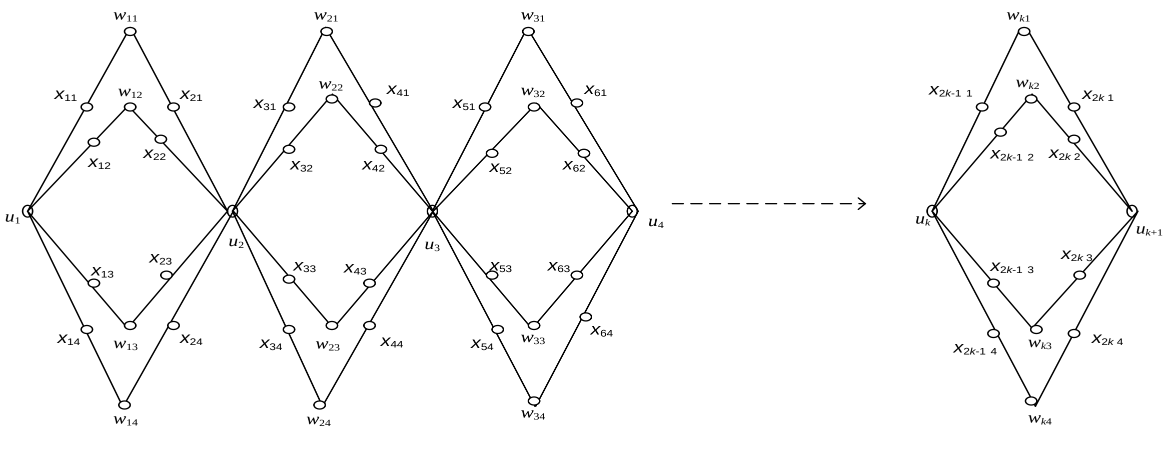

Figure 5: The graph (2, k) C_8.

The number of edges" q"= 16k

Define $\phi : V(G) \rightarrow \{0, 1, 2 \ldots q\}$ as following:

$\phi(u_i) = 8i - 8$, $i = 1, 2, 3 \ldots k+1$

$\phi(w_{ij}) = 8i - 2j + 1$, $i = 1, 2, 3 \ldots k$, $j = 1, 2, 3, 4$

$\phi(x_{ij}) = q - 4i - j + 5$, $i = 1, 2, 3 \ldots 2k$ for all $j = 1, 2, 3, 4$

From the definition of ϕ we find:

• $\forall\ v \in V(G)$: $V(G)$ is a set contains all the vertices of the graph $G = (2, k)\ C_8$-snake

$$\exists \max_{v \in V(G)} \phi(v) = \max \{ \max_{1 \le i \le k+1} (8i - 8), \max_{\substack{1 \le i \le k, \\ 1 \le j \le 4}} (8i - 2j + 1), \max_{\substack{1 \le i \le 2k, \\ 1 \le j \le 4}} (q - 4i - j + 5) \}$$

$$= \max \{ \max \{ 0, 8, 16 \ldots 8k\}, \max_{1 \le j \le 4} \{9 - 2j, 17 - 2j \ldots 8k - 2j + 1\},$$

$$\max_{1 \le j \le 4} \{q - j + 1, q - j - 3 \ldots q - j - 8k + 5\}\}$$

$$= \max\{8k, \max\{7, 5, 3, 1, 15, 13, 11, 9 \ldots 8k - 1, 8k - 3, 8k - 5, 8k - 7\}, \max\{q, q-1, q - 2, q - 3 \ldots q - 8k + 4, q - 8k + 3, q - 8k + 2, q - 8k + 1\}\}$$

$$= \max\{8k, 8k - 1, q\}$$
$$= q$$

Hence, we find that :−

$$\phi(v) \in \{ 0, 1, 2 \ldots q - 1, q \}$$

• It's obvious that $\forall\ u, v \in V(G)\ \exists\ \phi(u) \neq \phi(v) \in \{0, 1, 2 \ldots q - 1, q\}$

$\therefore$ The function ϕ is one-to-one mapping from the vertex set of G "$V(G)$" to the set $\{ 0, 1, 2 \ldots q \}$

• Now, we want to show that the labels of the edges of G belong to the set $\{1, 2 \ldots q \}$ and that's as following:

The range of $| \phi(x_{i^*1}) - \phi(u_i) | = \{q - 4i^* - 8i + 12$, $i^* = 1, 3, 5 \ldots 2k - 1$, $i = 1, 2 \ldots k \}$

$= \{ q, q - 16, q - 32 \ldots q - 16k + 16 \}$, $q = 16k$

$= \{ q, q - 16, q - 32 \ldots 16\}$

The range of $| \phi(x_{i^*1}) - \phi(w_{i1}) | = \{q - 4i^* - 8i + 5$, $i^* = 1, 3, 5 \ldots 2k - 1$, $i = 1, 2 \ldots k \}$

$= \{ q - 7, q - 23 \ldots q - 16k + 9 \}$, $q = 16k$

$= \{ q - 7, q - 23 \ldots 9\}$

The range of $| \phi(x_{i^*1}) - \phi(w_{i1}) | = \{q - 4i^* - 8i + 5$, $i^* = 2, 4, 6 \ldots 2k$, $i = 1, 2 \ldots k \}$

$= \{ q - 11, q - 27 \ldots q - 16k + 5 \}$, $q = 16k$

$= \{ q - 11, q - 27 \ldots 5\}$

The range of $|\phi(x_{i^*1}) - \phi(u_{i+1})| = \{q - 4i^* - 8i + 4$, $i^* = 2, 4, 6 \dots 2k$, $i = 1, 2 \dots k\}$

$= \{q - 12, q - 28 \dots q - 16k + 4\}$, $q = 16k$

$= \{q - 12, q - 28 \dots 4\}$

The range of $|\phi(x_{i^*2}) - \phi(u_i)| = \{q - 4i^* - 8i + 11$, $i^* = 1, 3, 5 \dots 2k - 1$, $i = 1, 2 \dots k\}$

$= \{q - 1, q - 17 \dots q - 16k + 15\}$, $q = 16k$

$= \{q - 1, q - 17 \dots 15\}$

The range of $|\phi(x_{i^*2}) - \phi(w_{i2})| = \{q - 4i^* - 8i + 6$, $i^* = 1, 3, 5 \dots 2k - 1$, $i = 1, 2 \dots k\}$

$= \{q - 6, q - 22 \dots q - 16k + 10\}$, $q = 16k$

$= \{q - 6, q - 22 \dots 10\}$

The range of $|\phi(x_{i^*2}) - \phi(w_{i2})| = \{q - 4i^* - 8i + 6$, $i^* = 2, 4, 6 \dots 2k$, $i = 1, 2 \dots k\}$

$= \{q - 10, q - 26 \dots q - 16k + 6\}$, $q = 16k$

$= \{q - 10, q - 26 \dots 6\}$

The range of $|\phi(x_{i^*2}) - \phi(u_{i+1})| = \{q - 4i^* - 8i + 3$, $i^* = 2, 4, 6 \dots 2k$, $i = 1, 2 \dots k\}$

$= \{q - 13, q - 29 \dots q - 16k + 3\}$, $q = 16k$

$= \{q - 13, q - 29 \dots 3\}$

The range of $|\phi(x_{i^*3}) - \phi(u_i)| = \{q - 4i^* - 8i + 10$, $i^* = 1, 3, 5 \dots 2k - 1$, $i = 1, 2 \dots k\}$

$= \{q - 2, q - 18 \dots q - 16k + 14\}$, $q = 16k$

$= \{q - 2, q - 18 \dots 14\}$

The range of $|\phi(x_{i^*3}) - \phi(w_{i3})| = \{q - 4i^* - 4i + 7$, $i^* = 1, 3, 5 \dots 2k - 1$, $i = 1, 2 \dots k\}$

$= \{q - 5, q - 21 \dots q - 16k + 11\}$, $q = 16k$

$= \{q - 5, q - 21 \dots 11\}$

The range of $|\phi(x_{i^*3}) - \phi(w_{i3})| = \{q - 4i^* - 8i + 7$, $i^* = 2, 4, 6 \dots 2k$, $i = 1, 2 \dots k\}$

$= \{q - 9, q - 25 \dots q - 16k + 7\}$, $q = 16k$

$= \{q - 9, q - 25 \dots 7\}$

The range of $|\phi(x_{i^*3}) - \phi(u_{i+1})| = \{q - 4i^* - 8i + 2$, $i^* = 2, 4, 6 \dots 2k$, $i = 1, 2 \dots k\}$

$= \{q - 14, q - 30 \dots q - 16k + 2\}$, $q = 16k$

$= \{ q - 14, q - 30 \ldots 2\}$

The range of $| \phi(x_{i^*4}) - \phi(u_i) | = \{q - 4i^* - 8i + 9$, $i^* = 1, 3, 5 \ldots 2k - 1$, $i = 1, 2 \ldots k \}$

$= \{ q - 3, q - 19 \ldots q - 16k + 13 \}$, $q = 16k$

$= \{ q - 3, q - 19 \ldots 13\}$

The range of $| \phi(x_{i^*4}) - \phi(w_{i4}) | = \{q - 4i^* - 8i + 8$, $i^* = 1, 3, 5 \ldots 2k - 1$, $i = 1, 2 \ldots k \}$

$= \{ q - 4, q - 20 \ldots q - 16k + 12 \}$, $q = 16k$

$= \{ q - 4, q - 20 \ldots 12\}$

The range of $| \phi(x_{i^*4}) - \phi(w_{i4}) | = \{q - 4i^* - 8i + 8$, $i^* = 2, 4, 6 \ldots 2k$, $i = 1, 2 \ldots k \}$

$= \{ q - 8, q - 24 \ldots q - 16k + 8 \}$, $q = 16k$

$= \{ q - 8, q - 24 \ldots 8\}$

The range of $| \phi(x_{i^*4}) - \phi(u_{i+1}) | = \{q - 4i^* - 8i + 1$, $i^* = 2, 4, 6 \ldots 2k$, $i = 1, 2 \ldots k \}$

$= \{ q - 15, q - 31 \ldots q - 16k + 1 \}$, $q = 16k$

$= \{ q - 15, q - 31 \ldots 1\}$

Hence, $\{ | \phi(u) - \phi(v) | : u\,v \in E(G) \} = \{ 1, 2, 3 \ldots q\}$.

So the graph (2, k) C_8 -snake is graceful.

Example 4

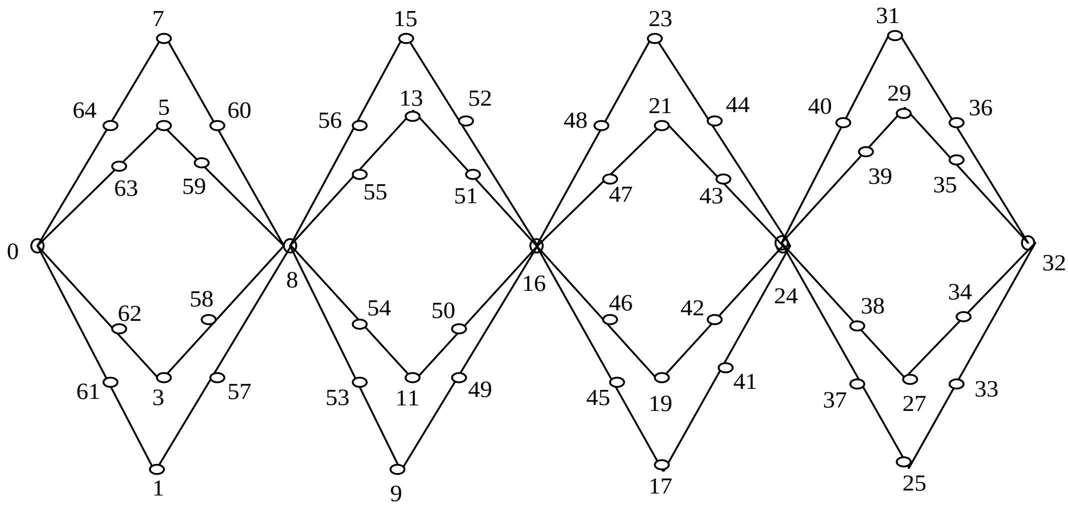

Figure 6: The graceful labeling of the graph (2, 4) C_8-snake

Theorem 3: The graph (3, k) C_8-snake is graceful.

Proof

Let u_i, where i = 1, 2, 3… k + 1, & w_{ij} where i = 1, 2, 3… k, j = 1, 2, 3, 4, 5, 6, & x_{ij} where i = 1, 2, 3… 2k, j = 1, 2, 3, 4, 5, 6 are the vertices of (3, k) C_8-snake, such that w_{ij} is put between u_i and u_{i+1}, x_{pj} is put between w_{ij} and u_i where i = 1, 2, 3…k, p is odd(p = 1, 3…2k - 1), j = 1, 2, 3, 4, 5, 6, x_{pj} is put between w_{ij} and u_{i+1} where i = 1, 2, 3…k, p is even(p = 2, 4…2k), j = 1, 2, 3, 4, 5, 6, w_{ij} is above $w_{i(j+1)}$ where j = 1, 2, 3, 4, 5, 6, x_{ij} is above $x_{i(j+1)}$ where j = 1, 2, 3, 4, 5, 6, the graph (3, k) C_8-snake has number of edges "q" 24k, as shown in the next figure.

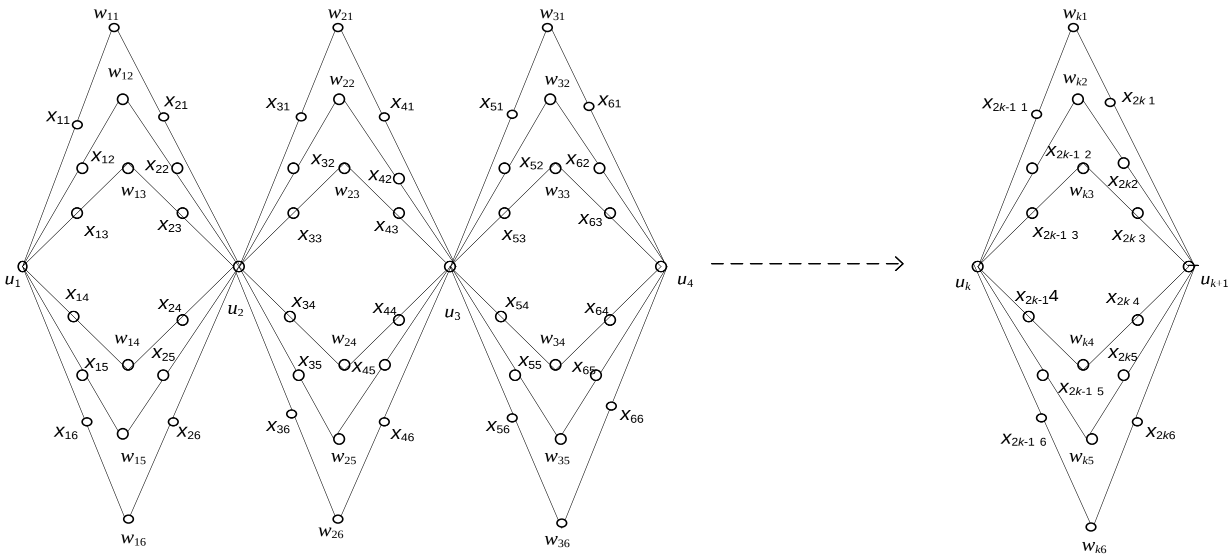

Figure 7: The graph (3, k) C_8.

The number of edges" q"= 24k

Define ϕ: $V(G)\rightarrow$ {0, 1, 2… q} as following:

$\phi(u_i)$ = 12i – 12 , i = 1, 2, 3… k+1

$\phi(w_{ij})$ = 12i – 2j + 1 , i = 1, 2, 3… k , j =1, 2, 3, 4, 5, 6

$\phi(x_{ij})$ = q – 6i – j + 7 , i = 1, 2, 3… 2k for all j =1, 2, 3, 4, 5, 6

From the definition of ϕ we find:

• $\forall\ v \in V(G)$: $V(G)$ is a set contains all the vertices of the graph G = (3, k) C_8-snake

$$\exists \max_{v\in V(G)} \phi(v) = \max \{ \max_{1\le i\le k+1}(12i-12),\ \max_{\substack{1\le i\le k,\\ 1\le j\le 6}}(12i-2j+1),\ \max_{\substack{1\le i\le 2k,\\ 1\le j\le 6}}(q-6i-j+7)\}$$

$$= \max \{ \max \{ 0, 12, 24 \dots 12k\},\ \max_{1\le j\le 4} \{13-2j, 25-2j \dots 12k-2j+1\},$$

$$\max_{1\le j\le 6} \{q-j+1, q-j-5 \dots q-j-12k+7\}\}$$

$$= \max\{12k, 12k-1, q\}$$
$$= q$$

Hence, we find that :−

$$\phi(v) \in \{ 0, 1, 2 \dots q-1, q \}$$

• It's obvious that $\forall\ u, v \in V(G)\ \exists\ \phi(u) \neq \phi(v) \in \{0, 1, 2 \ldots q-1, q\}$

$\therefore$ The function ϕ is one-to-one mapping from the vertex set of G "$V(G)$" to the set $\{0, 1, 2 \ldots q\}$

• Now, we want to show that the labels of the edges of G belong to the set $\{1, 2 \ldots q\}$ and that's as following:

The range of $|\phi(x_{i^*1}) - \phi(u_i)| = \{q - 6i^* - 12i + 18 \quad , i^* = 1, 3, 5 \ldots 2k - 1, i = 1, 2 \ldots k\}$

$= \{q, q - 24, q - 48 \ldots q - 24k + 24\} \quad , q = 24k$

$= \{q, q - 24, q - 48 \ldots 24\}$

The range of $|\phi(x_{i^*1}) - \phi(w_{i1})| = \{q - 6i^* - 12i + 7 \quad , i^* = 1, 3, 5 \ldots 2k - 1, i = 1, 2 \ldots k\}$

$= \{q - 11, q - 35 \ldots q - 24k + 13\} \quad , q = 24k$

$= \{q - 11, q - 35 \ldots 13\}$

The range of $|\phi(x_{i^*1}) - \phi(w_{i1})| = \{q - 6i^* - 12i + 7 \quad , i^* = 2, 4, 6 \ldots 2k, i = 1, 2 \ldots k\}$

$= \{q - 17, q - 41 \ldots q - 24k + 7\} \quad , q = 24k$

$= \{q - 17, q - 41 \ldots 7\}$

The range of $|\phi(x_{i^*1}) - \phi(u_{i+1})| = \{q - 6i^* - 12i + 6 \quad , i^* = 2, 4, 6 \ldots 2k, i = 1, 2 \ldots k\}$

$= \{q - 18, q - 42 \ldots q - 24k + 6\} \quad , q = 24k$

$= \{q - 18, q - 42 \ldots 6\}$

The range of $|\phi(x_{i^*2}) - \phi(u_i)| = \{q - 6i^* - 12i + 17 \quad , i^* = 1, 3, 5 \ldots 2k - 1, i = 1, 2 \ldots k\}$

$= \{q - 1, q - 25 \ldots q - 24k + 23\} \quad , q = 24k$

$= \{q - 1, q - 25 \ldots 23\}$

The range of $|\phi(x_{i^*2}) - \phi(w_{i2})| = \{q - 6i^* - 12i + 8 \quad , i^* = 1, 3, 5 \ldots 2k - 1, i = 1, 2 \ldots k\}$

$= \{q - 10, q - 34 \ldots q - 24k + 14\} \quad , q = 24k$

$= \{q - 10, q - 34 \ldots 14\}$

The range of $|\phi(x_{i^*2}) - \phi(w_{i2})| = \{q - 6i^* - 12i + 8 \quad , i^* = 2, 4, 6 \ldots 2k, i = 1, 2 \ldots k\}$

$= \{q - 16, q - 40 \ldots q - 24k + 8\} \quad , q = 24k$

$= \{q - 16, q - 40 \ldots 8\}$

The range of $|\phi(x_{i^*2}) - \phi(u_{i+1})| = \{q - 6i^* - 12i + 5 \quad , i^* = 2, 4, 6 \ldots 2k, i = 1, 2 \ldots k\}$

$= \{q - 19, q - 34 \ldots q - 24k + 5\} \quad , q = 24k$

$= \{q - 19, q - 34 \ldots 5\}$

The range of $|\phi(x_{i^*3}) - \phi(u_i)| = \{q - 6i^* - 12i + 16 \quad , i^* = 1, 3, 5 \ldots 2k - 1 , i = 1, 2 \ldots k\}$

$= \{q - 2, q - 26 \ldots q - 24k + 22\} \quad , q = 24k$

$= \{q - 2, q - 26 \ldots 22\}$

The range of $|\phi(x_{i^*3}) - \phi(w_{i3})| = \{q - 6i^* - 12i + 9 \quad , i^* = 1, 3, 5 \ldots 2k - 1 , i = 1, 2 \ldots k\}$

$= \{q - 9, q - 33 \ldots q - 24k + 15\} \quad , q = 24k$

$= \{q - 9, q - 33 \ldots 15\}$

The range of $|\phi(x_{i^*3}) - \phi(w_{i3})| = \{q - 6i^* - 12i + 9 \quad , i^* = 2, 4, 6 \ldots 2k , i = 1, 2 \ldots k\}$

$= \{q - 15, q - 39 \ldots q - 24k + 9\} \quad , q = 24k$

$= \{q - 15, q - 39 \ldots 9\}$

The range of $|\phi(x_{i^*3}) - \phi(u_{i+1})| = \{q - 6i^* - 12i + 4 \quad , i^* = 2, 4, 6 \ldots 2k , i = 1, 2 \ldots k\}$

$= \{q - 20, q - 44 \ldots q - 24k + 4\} \quad , q = 24k$

$= \{q - 20, q - 44 \ldots 4\}$

The range of $|\phi(x_{i^*4}) - \phi(u_i)| = \{q - 6i^* - 12i + 15 \quad , i^* = 1, 3, 5 \ldots 2k - 1 , i = 1, 2 \ldots k\}$

$= \{q - 3, q - 27 \ldots q - 24k + 21\} \quad , q = 24k$

$= \{q - 3, q - 27 \ldots 21\}$

The range of $|\phi(x_{i^*4}) - \phi(w_{i4})| = \{q - 6i^* - 12i + 10 \quad , i^* = 1, 3, 5 \ldots 2k - 1 , i = 1, 2 \ldots k\}$

$= \{q - 8, q - 32 \ldots q - 24k + 16\} \quad , q = 24k$

$= \{q - 8, q - 32 \ldots 16\}$

The range of $|\phi(x_{i^*4}) - \phi(w_{i4})| = \{q - 6i^* - 12i + 10 \quad , i^* = 2, 4, 6 \ldots 2k , i = 1, 2 \ldots k\}$

$= \{q - 14, q - 38 \ldots q - 24k + 10\} \quad , q = 24k$

$= \{q - 14, q - 38 \ldots 10\}$

The range of $|\phi(x_{i^*4}) - \phi(u_{i+1})| = \{q - 6i^* - 12i + 3 \quad , i^* = 2, 4, 6 \ldots 2k , i = 1, 2 \ldots k\}$

$= \{q - 21, q - 45 \ldots q - 24k + 3\} \quad , q = 24k$

$= \{q - 21, q - 45 \ldots 3\}$

The range of $|\phi(x_{i^*5}) - \phi(u_i)| = \{q - 6i^* - 12i + 14 \quad , i^* = 1, 3, 5 \ldots 2k - 1 , i = 1, 2 \ldots k\}$

$= \{q - 4, q - 28 \ldots q - 24k + 20\} \quad , q = 24k$

$= \{q - 4, q - 28 \ldots 20\}$

The range of $| \phi(x_{i^*5}) - \phi(w_{i5}) | = \{q - 6i^* - 12i + 11 \quad , i^* = 1, 3, 5 \ldots 2k - 1 , i = 1, 2 \ldots k \}$

$= \{ q - 7, q - 31 \ldots q - 24k + 17 \} \quad , q = 24k$

$= \{ q - 7, q - 31 \ldots 17\}$

The range of $| \phi(x_{i^*5}) - \phi(w_{i5}) | = \{q - 6i^* - 12i + 11 \quad , i^* = 2, 4, 6 \ldots 2k , i = 1, 2 \ldots k \}$

$= \{ q - 13, q - 37 \ldots q - 24k + 11 \} \quad , q = 24k$

$= \{ q - 13, q - 37 \ldots 11\}$

The range of $| \phi(x_{i^*5}) - \phi(u_{i+1}) | = \{q - 6i^* - 12i + 2 \quad , i^* = 2, 4, 6 \ldots 2k , i = 1, 2 \ldots k \}$

$= \{ q - 22, q - 46 \ldots q - 24k + 2 \} \quad , q = 24k$

$= \{ q - 22, q - 46 \ldots 2\}$

The range of $| \phi(x_{i^*6}) - \phi(u_i) | = \{q - 6i^* - 12i + 13 \quad , i^* = 1, 3, 5 \ldots 2k - 1 , i = 1, 2 \ldots k \}$

$= \{ q - 5, q - 29 \ldots q - 24k + 19 \} \quad , q = 24k$

$= \{ q - 5, q - 29 \ldots 19\}$

The range of $| \phi(x_{i^*6}) - \phi(w_{i6}) | = \{q - 6i^* - 12i + 12 \quad , i^* = 1, 3, 5 \ldots 2k - 1 , i = 1, 2 \ldots k \}$

$= \{ q - 6, q - 30 \ldots q - 24k + 18 \} \quad , q = 24k$

$= \{ q - 6, q - 30 \ldots 18\}$

The range of $| \phi(x_{i^*6}) - \phi(w_{i6}) | = \{q - 6i^* - 12i + 12 \quad , i^* = 2, 4, 6 \ldots 2k , i = 1, 2 \ldots k \}$

$= \{ q - 12, q - 36 \ldots q - 24k + 12 \} \quad , q = 24k$

$= \{ q - 12, q - 36 \ldots 12\}$

The range of $| \phi(x_{i^*6}) - \phi(u_{i+1}) | = \{q - 6i^* - 12i + 1 \quad , i^* = 2, 4, 6 \ldots 2k , i = 1, 2 \ldots k \}$

$= \{ q - 23, q - 47 \ldots q - 24k + 1 \} \quad , q = 24k$

$= \{ q - 23, q - 47 \ldots 1\}$

Hence, $\{ | \phi(u) - \phi(v) | : u\,v \in E(G) \} = \{ 1, 2, 3 \ldots q\}$.

So the graph $(3, k)$ C_8-snake is graceful.

Example 5

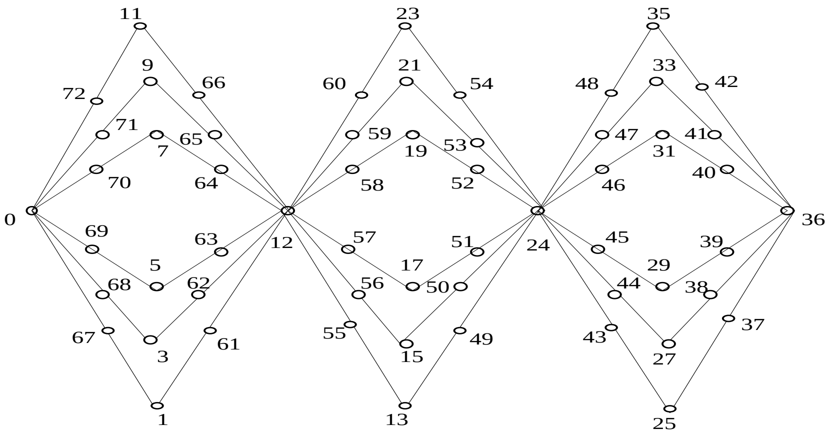

Figure 8: The graceful labeling of the graph (3, 3) C_8-snake

Theorem 4: The graph (m, k) C_8-snake is graceful.

Proof

Let u_i, where $i = 1, 2, 3\ldots k + 1$, & w_{ij} where $i = 1, 2, 3\ldots 2m$, & x_{ij} where $i = 1, 2, 3\ldots 2k$, $j = 1, 2, 3\ldots 2m$ are the vertices of (m, k) C_8-snake, such that w_{ij} is put between u_i and u_{i+1}, x_{pj} is put between w_{ij} and u_i where $i = 1, 2, 3\ldots k$, p is odd($p = 1, 3\ldots 2k - 1$), $j = 1, 2, 3\ldots 2m$, x_{pj} is put between w_{ij} and u_{i+1} where $i = 1, 2, 3\ldots k$, p is even($p = 2, 4\ldots 2k$), $j = 1, 2, 3\ldots 2m$, w_{ij} is above $w_{i(j+1)}$ where $j = 1, 2, 3\ldots 2m$, x_{ij} is above $x_{i(j+1)}$ where $j = 1, 2, 3...2m$, the graph (m, k) C_8-snake has number of edges "q" $8mk$, as shown in the next figure.

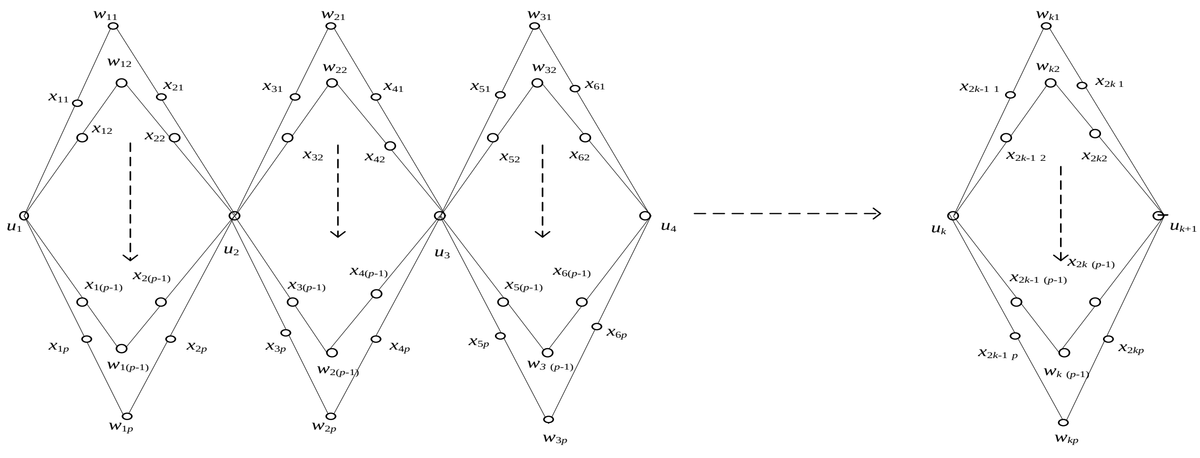

Figure 9: The graphs (m, k) C_8-snake

The number of edges" q"$= 8mk$, $m = 1, 2, 3 \ldots$, $p = 2m$

Define $\phi : V(G) \to \{0, 1, 2\ldots q\}$ as following:

$\phi(u_i) = 4m(i-1)$, $i = 1, 2, 3\ldots k+1$

$\phi(w_{ij}) = 4mi - 2j + 1$, $i = 1, 2, 3\ldots k$, $j = 1, 2, 3\ldots 2m$

$\phi(x_{ij}) = q - 2m(i-1) - j + 1$, $i = 1, 2, 3\ldots 2k$, $j = 1, 2, 3\ldots 2m$

From the definition of ϕ we find:

• $\forall\, v \in V(G)$: $V(G)$ is a set contains all the vertices of the graph $G = (m, k)$ C_8-snake

$$\exists \max_{v\in V(G)} \phi(v) = \max\{ \max_{1\le i\le k+1}(4mi - 4m), \max_{\substack{1\le i\le k,\\ 1\le j\le 2m}}(4mi - 2j + 1), \max_{\substack{1\le i\le 2k,\\ 1\le j\le 2m}}(q - 2m(i-1) - j + 1)\}$$

$$= \max\{ \max\{0, 4m, 8m \ldots 4mk\}, \max_{1\le j\le 2m}\{4m - 2j + 1, 8m - 2j + 1 \ldots 4mk - 2j + 1\}, \max_{1\le j\le 2m}\{q - j + 1, q - j - 4m + 1 \ldots q - j - 4mk + 2m + 1\}\}$$

$= \max\{4mk, 4mk - 1, q\}$
$= q$

Hence, we find that :−

$\phi(v) \in \{0, 1, 2 \ldots q-1, q\}$

• It's obvious that $\forall\, u, v \in V(G)\ \exists\ \phi(u) \neq \phi(v) \in \{0, 1, 2 \ldots q-1, q\}$

∴ The function ϕ is one-to-one mapping from the vertex set of G "$V(G)$" to the set $\{0, 1, 2 \ldots q\}$

• Now, we want to show that the labels of the edges of G belong to the set $\{1, 2 \ldots q\}$ and that's as following:

The range of $|\phi(x_{i^*j}) - \phi(u_i)| = \{q - 2mi^* - 4mi - j + 6m + 1$, $i^* = 1, 3, 5 \ldots 2k - 1$, $i = 1, 2 \ldots k$, $j = 1, 2 \ldots 2m\}$

$= \{q - j + 1, q - j - 8m + 1 \ldots q - j - 8mk + 8m + 1$, $j = 1, 2 \ldots 2m\}$

The range of $|\phi(x_{i^*j}) - \phi(w_{ij})| = \{q - 2mi^* - 4mi + j + 2m$, $i^* = 1, 3, 5 \ldots 2k - 1$, $i = 1, 2 \ldots k$, $j = 1, 2 \ldots 2m\}$

$= \{q + j - 4m, q + j - 12m \ldots q + j - 8mk + 4m$, $j = 1, 2 \ldots 2m\}$

The range of $|\phi(x_{i^*j}) - \phi(w_{ij})| = \{q - 2mi^* - 4mi + j + 2m$, $i^* = 2, 4, 6 \ldots 2k - 1$, $i = 1, 2 \ldots k$, $j = 1, 2 \ldots 2m\}$

$= \{q + j - 6m, q + j - 14m \ldots q + j - 8mk + 2m$, $j = 1, 2 \ldots 2m\}$

The range of $|\phi(x_{i^*j}) - \phi(u_{i+1})| = \{q - 2mi^* - 4mi - j + 2m + 1$, $i^* = 2, 4, 6 \ldots 2k - 1$, $i = 1, 2 \ldots k$, $j = 1, 2 \ldots 2m\}$

$= \{ q - j - 6m + 1 , q - j - 14m +1 \ldots q + j - 8mk + 2m + 1 , j = 1, 2 \ldots 2m \}$

Hence, $\{ | \phi(u) - \phi(v) | : u\, v \in E(G) \} = \{ 1, 2, 3 \ldots q\}$.

So the graph (m, k) C_8 -snake is graceful.

Example 6

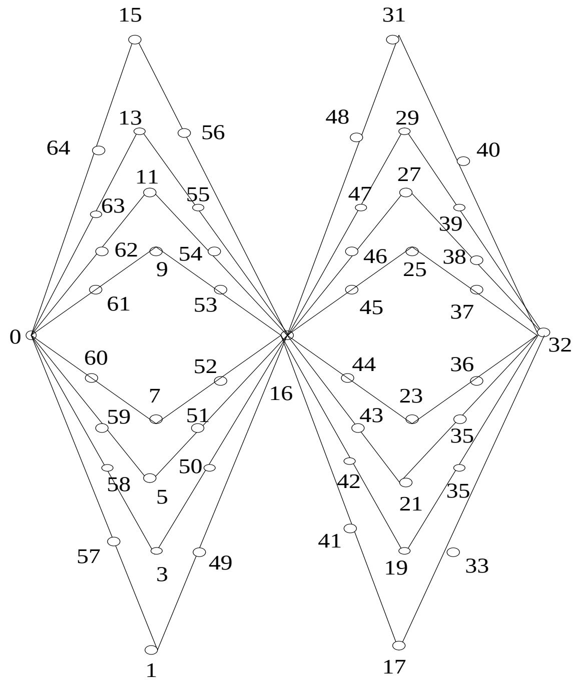

Figure 10: The graceful labeling of the graph (4, 2) C_8-snake

Conclusion:

In this paper we introduced the graceful labeling of the graphs kC_4- snake, (2, k) C_4-snake, (3, k) C_4-snake, finally we introduced the graceful labeling of the graph (m, k) C_4-snake**.** we introduced the graceful labeling of the graphs kC_8- snake, (2, k) C_8-snake, (3, k) C_8-snake, finally we introduced the graceful labeling of the graph (m, k) C_8-snake

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